Generalized entropies and the transformation group of superstatistics

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Superstatistics describes statistical systems that behave like superpositions of different inverse temperatures β, so that the probability distribution is $p(\epsilon_i) \propto \int_0^\infty f(\beta) e^{-\beta\epsilon_i} d\beta$, where the "kernel" $f(\beta)$ is nonnegative and normalized [$\int f(\beta) d\beta = 1$]. We discuss the relation between this distribution and the generalized entropic form $S = \sum_i s(p_i)$. The first three Shannon–Khinchin axioms are assumed to hold. It then turns out that for a given distribution there are two different ways to construct the entropy. One approach uses escort probabilities and the other does not; the question of which to use must be decided empirically. The two approaches are related by a duality. The thermodynamic properties of the system can be quite different for the two approaches. In that connection, we present the transformation laws for the superstatistical distributions under macroscopic state changes. The transformation group is the Euclidean group in one dimension.

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Superstatistics (1) has been introduced as a way of systematically handling statistical systems that can be seen as superpositions of various Boltzmann distributions. One may think of a superstatistical system on a microscopic and a macroscopic scale. At the microscopic scale, the system relaxes toward thermodynamic equilibrium and follows a single Boltzmann statistic with a well-defined *local* inverse temperature β—i.e., the local probability of finding the system at some energy ϵ is proportional to $\exp(-\beta\epsilon)$. On the macroscopic scale, the local inverse temperatures fluctuate, with the fluctuations governed by $f(\beta)$.* Following ref. 1, the generalized superstatistical Boltzmann factor reads

$$B(\epsilon) = \int_0^\infty d\beta f(\beta) e^{-\beta\epsilon}, \qquad [1]$$

where $f(\beta)$ is called the *superstatistical kernel*. The kernel is nonnegative and normalized, i.e., $\int_0^\infty d\beta f(\beta) = 1$. The probability of finding the system in one of W discrete (for example, energy) states, $\epsilon = \{\epsilon_i\}_{i=1}^W$, is

$$p_i = \frac{1}{Z} B(\epsilon_i), \qquad Z = \sum_{i=1}^W B(\epsilon_i), \qquad [2]$$

Z being the superstatistical generalization of the partition function. The expectation value

$$U = \sum_{i=1}^W p_i \epsilon_i \qquad [3]$$

is the internal energy of the system.

Superstatistics has found many applications ranging from hydrodynamic turbulence (2–4), complex networks (5, 6), and pattern formation (7) to finance (8, 9). Besides these practical aspects, superstatistics has the potential for providing a structural foundation for non-Boltzmann statistical mechanics.

Different methods have been proposed for establishing the mathematical methodology of a kind of statistical mechanics compatible with superstatistics (10–14). The methods that use generalized entropic forms are the Tsallis Souza (TS) approach (10) and one described in refs. 13 and 14, which we refer to as the Hanel Thurner (HT) approach. Both use a maximum entropy principle (MEP) to *reconstruct* generalized entropies having the structure

$$S[\mathbf{p}] = \sum_{i=1}^W s(p_i) \qquad [4]$$

from given superstatistical distribution functions. Here $\mathbf{p}$ stands for the set of probabilities $\{p_i\}_{i=1}^W$. The sum structure is chosen in analogy to Boltzmann–Gibbs (BG) entropy, where $s_{BG}(x) = -x\log(x)$. The function s is continuous and concave and has $s(0) = 0$. These properties correspond to the first three Shannon–Khinchin axioms† (15, 16). In this paper, we address the question of whether the TS approach is the unique way to reconstruct entropies for superstatistical systems. The answer is no—there exist possible alternatives. In fact, if we did not have the first Khinchin axiom, we would have infinitely many possibilities. We show however, that assuming this axiom, we have only two possibilities remaining. The axiom requires that the entropy be a continuous function of the probabilities $\mathbf{p}$ only (compare footnote †). It turns out that one possibility corresponds to the TS (10) and the other to the HT approach (13, 14). We show that the HT case cannot be seen as a simple limit of the TS case. For a large class of distribution functions such a limit yields the BG case, not HT in general. The thermodynamic properties of the entropies corresponding to the two approaches will in general be quite different. Which approach is the correct one to use in each case has to be decided empirically. Superstatistics—as originally presented—does not make any assumptions about how the superstatistical Boltzmann factor B or the kernel f transform under thermodynamic state changes. On the macroscopic scale, a superstatistical system can be described by a macrostate σ, in complete analogy to a classical thermodynamic system. These states are characterized by a set of macrovariables, e.g., internal energy, volume, temperatures, $\sigma = \{U, V, T, \ldots\}$. We present the transformation group of the superstatistical kernel f under changes from one state to another, $\sigma \to \sigma'$.

Entropy Reconstruction from Distribution Functions

The usual procedure when using the MEP is to maximize a given entropy under given constraints, represented by Lagrange multipliers:

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*Superstatistical systems are usually not in equilibrium but in stationary nonequilibrium. E.g., think of a gas or liquid between two plates at different temperatures T_1 and T_2.

†Shannon–Khinchin axioms: (*i*) Entropy is a continuous function of the probabilities p_i only, i.e., s should not explicitly depend on any other parameters. (*ii*) Entropy is maximal for the equidistribution $p_i = 1/W$—from which the concavity of s follows. (*iii*) Adding to a system a state numbered $W+1$ with $p_{W+1} = 0$ does not change the entropy of the system—from which $s(0) = 0$ follows.

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$$\Phi[\mathbf{p}] = S[\mathbf{p}] - \alpha\left(\sum_i p_i - 1\right) - \beta\left(\sum_i p_i \epsilon_i - U\right). \quad [5]$$

Given $s(p_i)$ and U, this maximization yields the distribution function p_i, along with the values of the Lagrange multipliers α and β.

In the case of entropy reconstruction, the distribution function of a superstatistical system is given. The system is in a specific physical state σ (which does not change during the reconstruction). The various quantities such as $p_i^{(\sigma)}$, the internal energy $U^{(\sigma)}$, etc. are given and are labeled by (σ). Because $p^{(\sigma)}$ is a superstatistical distribution, we have $p_i^{(\sigma)} = B(\epsilon^{(\sigma)})/Z^{(\sigma)}$. The quantity $s(p_i)$ and the entropy can now be computed for the state σ.

Entropy Reconstruction—the HT Approach. We begin with the simpler case and demonstrate the entropy reconstruction using the HT approach (13, 14), which starts by maximizing the functional of Eq. **5**. At this point, we use the available information about the specific state σ. The condition

$$\partial\Phi[\mathbf{p}^{(\sigma)}]/\partial p_i^{(\sigma)} = 0 \quad [6]$$

leads to

$$s'_{\mathrm{HT}}(p_i^{(\sigma)}) = \alpha^{(\sigma)} + \beta^{(\sigma)}\epsilon_i^{(\sigma)}. \quad [7]$$

Because $B^{(\sigma)}(\epsilon)$ is a monotonic function of ϵ, the inverse function L defined by

$$L\left(\frac{B^{(\sigma)}(\epsilon)}{Z^{(\sigma)}}\right) = \epsilon \quad [8]$$

exists. Note that, for the specific state σ, $Z^{(\sigma)}$ is just a positive real number. Otherwise the inverse would be hard to calculate. It follows immediately that

$$L(p_j^{(\sigma)}) = \epsilon_j^{(\sigma)} \quad [9]$$

for all j. Eqs. **7** and **9** yield the differential equation

$$s'_{\mathrm{HT}}(p_i^{(\sigma)}) = \alpha^{(\sigma)} + \beta^{(\sigma)}L(p_i^{(\sigma)}), \quad [10]$$

which allows us to reconstruct the entropic form by integration:

$$s_{\mathrm{HT}}(x) = \alpha^{(\sigma)}x + \beta^{(\sigma)}\int_0^x dyL(y). \quad [11]$$

We have now derived an entropic form s_{HT} which, when used in the MEP, Eq. **5**, reproduces $p_i^{(\sigma)}$, $\alpha^{(\sigma)}$, and $\beta^{(\sigma)}$ as a solution, given the state characterized by $U^{(\sigma)}$ and $\epsilon_i^{(\sigma)}$. [In refs. 13 and 14, $\Lambda(x) = -\alpha - \beta L(x)$ is called the generalized logarithm.] As a result of the first Khinchin axiom, this entropic form s_{HT} characterizes the superstatistical system in general—i.e., not only for the state σ but also for the other possible states σ'—corresponding to different values of $U^{(\sigma')}$ or $\epsilon_i^{(\sigma')}$. This fact has consequences for the thermodynamical treatment of the system. In particular, transformation laws for the superstatistical Boltzmann factor B and the kernel f under changes of the state σ can be derived, as will be shown below.

Entropy Reconstruction with Escort Distributions—the TS Approach. We now generalize the above idea by modifying the energy constraint. We compute a different "energy" U^* by using a different set of "probabilities." These probabilities can be written in the general form

$$P_i = \frac{u(p_i)}{\sum_j u(p_j)}. \quad [12]$$

The quantity P_i is often called *escort probability*. The escort energy U^* is identical with the internal energy U if and only if $u(p_i) = p_i$. We do not attempt to provide a physical interpretation of the escort probabilities. We start with the functional

$$\Phi[\mathbf{p}] = S[\mathbf{p}] - \alpha\left(\sum_i p_i - 1\right) - \beta\left(\sum_i P_i \epsilon_i - U^*\right). \quad [13]$$

For taking the derivative of P_i, we define the following notation using a function Q of two variables:

$$\begin{aligned} u(p_i) &\equiv Q[x,y]|_{x=p_i;y=s(p_i)}, \\ u_1(p_i) &\equiv \frac{\partial}{\partial x}Q[x,y]|_{x=p_i;y=s(p_i)}, \\ u_2(p_i) &\equiv \frac{\partial}{\partial y}Q[x,y]|_{x=p_i;y=s(p_i)}. \end{aligned} \quad [14]$$

Maximizing the functional in Eq. **13**, we have $\partial\Phi/\partial p_i = 0$, leading to

$$s'(p_i) = \alpha + \beta^*(\epsilon_i - U^*)[u_1(p_i) + u_2(p_i)s'(p_i)], \quad [15]$$

where β^* is the rescaled parameter $\beta/\sum_i u(p_i)$. Solving for s', we get

$$s'(p_i) = \frac{\alpha + \beta^*(\epsilon_i - U^*)u_1(p_i)}{1 - \beta^*(\epsilon_i - U^*)u_2(p_i)}. \quad [16]$$

As in the HT approach, we consider from this point on a specific state σ of the superstatistical system and use the specific values $\alpha^{(\sigma)}, \beta^{(\sigma)}, \ldots$, for reconstructing the associated entropy s. However, for the sake of readability, we suppress (σ) whenever that cannot cause confusion. Using Eq. **9** as above, we insert $\epsilon_i = L(p_i)$ into Eq. **16** and get

$$s'(x) = \frac{\alpha + \beta^*[L(x) - U^*]u_1(x)}{1 - \beta^*[L(x) - U^*]u_2(x)}. \quad [17]$$

This differential equation in s can now be solved, so that, for any suitable function u, an entropic form s is obtained, which reproduces the superstatistical distribution function $p_i^{(\sigma)} = B^{(\sigma)}(\epsilon_i)/Z^{(\sigma)}$ under the MEP.

It is easy to see that such entropies could explicitly depend on U^*, except for the first Khinchin axiom, which rules out such dependence, dramatically reducing the possibilities for choosing u. In fact, the first axiom can hold only if both $u_1(x) = r_1$ and $u_2(x) = r_2$ are constants, implying $u(p_i) = r_1 p_i + r_2 s(p_i)$, which now allows us to absorb U^* into the Lagrange multipliers. By defining the constants

$$\hat{\alpha} = \frac{\alpha - \beta^* U^* r_1}{1 + \beta^* U^* r_2} \quad \text{and} \quad \hat{\beta} = \frac{\beta^* r_1}{1 + \beta^* U^* r_2}, \quad [18]$$

we can simplify Eq. **17** to

$$s'(x) = \frac{\hat{\alpha} + \hat{\beta}L(x)}{1 - \hat{\beta}\nu L(x)} \quad [19]$$

with $\nu = r_2/r_1$. This equation can now be solved by simple integration. Note that because P_i remains invariant if both r_1 and r_2 are rescaled by the same factor, we can use $r_1 = 1$ without loss of

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generality. With $u(p_i) = p_i + \nu s(p_i)$, the TS approach is recovered. Note that more general constraints than those in Eqs. **5** and **13** are ruled out by the first Khinchin axiom.

Two Entropies

We see that only two entropies remain, one obtained with ordinary constraints (HT approach) and the other by employing escort distributions in the functional of Eq. **13** (TS approach).

- **HT entropy**: The approach without escort distributions results in Eq. **11**

$$s_{\mathrm{HT}}(x) = \beta \int_0^x dy L(y) + \alpha x, \qquad [20]$$

  which yields the HT entropy (13, 14).
- **TS entropy**: The only possible entropy involving escort distributions results from Eq. **19**, which gives

$$s_{\mathrm{TS}}(x) = \int_0^x dy \frac{\hat{\alpha} + \hat{\beta} L(y)}{1 - \hat{\beta}\nu L(y)}, \qquad [21]$$

  the TS result for the entropy, and $P_i = [p_i + \nu s_{\mathrm{TS}}(p_i)] / \sum_j [p_j + \nu s_{\mathrm{TS}}(p_j)]$ for the constraint. In ref. 10, $\hat{\beta}$ is set equal to one.

Duality of the Two Approaches. Using Eqs. **20** and **21**, we have for the functional relation between the two entropies

$$s'_{\mathrm{TS}}(x) = \frac{\frac{1}{\beta}(\hat{\alpha}\beta - \alpha\hat{\beta}) + s'_{\mathrm{HT}}(x)}{(\frac{\beta}{\hat{\beta}} + \nu\alpha) - \nu s'_{\mathrm{HT}}(x)}. \qquad [22]$$

Suppose we reconstruct an entropic form s from state σ. We now use this form in the MEP for another state σ', which could, for example, be a state with a different internal energy. For both approaches, the HT and the TS, solving the MEP for state σ' leads to distribution functions of the form

$$p_i^{(\sigma')} = \frac{1}{Z^{(\sigma)}} B^{(\sigma)}(a + b\epsilon_i^{(\sigma')}), \qquad [23]$$

where, for the HT approach, we have

$$a = \frac{\alpha^{(\sigma')} - \alpha^{(\sigma)}}{\beta^{(\sigma)}} \quad \text{and} \quad b = \frac{\beta^{(\sigma')}}{\beta^{(\sigma)}}, \qquad [24]$$

and for the TS approach,

$$a = \frac{\hat{\alpha}^{(\sigma')} - \hat{\alpha}^{(\sigma)}}{\hat{\beta}^{(\sigma)}(1 + \nu\hat{\alpha}^{(\sigma')})} \quad \text{and} \quad b = \frac{\hat{\beta}^{(\sigma')}}{\hat{\beta}^{(\sigma)}} \frac{1 + \nu\hat{\alpha}^{(\sigma)}}{1 + \nu\hat{\alpha}^{(\sigma')}}. \qquad [25]$$

From Eqs. **24** and **25** and the requirements $\lim_{\nu\to 0} \hat{\alpha} = \alpha$ and $\lim_{\nu\to 0} \hat{\beta} = \beta$, we get

$$0 = \hat{\alpha}\beta - \alpha\hat{\beta} \quad \text{and} \quad 1 = \frac{\beta}{\hat{\beta}} + \nu\alpha, \qquad [26]$$

so that Eq. **22** simplifies to the relation

$$\frac{1}{s'_{\mathrm{TS}}(x)} - \frac{1}{s'_{\mathrm{HT}}(x)} = -\nu. \qquad [27]$$

Remarkably, α and β drop out. This relation can be expressed in terms of so-called generalized logarithms (g logarithms) for the HT and the TS approaches. Generalized logarithms $\Lambda(x)$ have been widely used in the context of generalized entropies, e.g., refs. 17 and 18. Λ is an increasing monotonic function with $\Lambda(1) = 0$ and $\Lambda'(1) = 1$. Eq. **27** suggests that, given $s'_{\mathrm{TS}}(x) = 0$ for some $x = x_0$, $s'_{\mathrm{HT}}(x_0)$ is also 0, and further $s''_{\mathrm{TS}}(x_0) = s''_{\mathrm{HT}}(x_0)$. If we define a constant $c \equiv -x_0 s''_{\mathrm{HT}}(x_0)$, the g logarithms associated with the generalized entropies can be written as

$$\Lambda_{\mathrm{HT}}(x) = -\frac{1}{c} s'_{\mathrm{HT}}(x_0 x), \qquad \Lambda_{\mathrm{TS}}(x) = -\frac{1}{c} s'_{\mathrm{TS}}(x_0 x). \qquad [28]$$

With these g logarithms, Eq. **27** becomes the *duality relation*

$$\Lambda^*_{\mathrm{HT}}(x) = \Lambda_{\mathrm{TS}}(x), \qquad [29]$$

where the duality operation * is defined by

$$\Lambda^* = \frac{1}{\frac{1}{\Lambda} + c\nu}. \qquad [30]$$

Note that Eq. **30** possesses a symmetry: The equation is invariant under interchanging Λ ($\Lambda \equiv \Lambda_{\mathrm{HT}}$) with Λ^* ($\Lambda^* \equiv \Lambda_{\mathrm{TS}}$) and simultaneously changing the sign of ν. Obviously, applying * a second time (exchanging $\Lambda_{\mathrm{HT}} \leftrightarrow \Lambda_{\mathrm{TS}}$ and $\nu \leftrightarrow -\nu$) yields the identity, as has to be the case for any duality[‡]. In Eq. **30**, we see that the unique value of ν, where Λ^* and Λ and hence s_{HT} and s_{TS} can coincide, is $\nu = 0$. On one hand, in the $\nu \to 0$ limit, the HT and the TS entropies are obviously identical. On the other hand, superstatistical systems are generalizations of BG statistics. It is therefore natural for the two approaches to coincide for the BG case.

Example: Duality and Power Laws. For a large class of distribution functions, the duality relation of the associated g logarithms includes the condition

$$\Lambda^*(x) = -\Lambda(1/x) \qquad [31]$$

given in ref. 17. More precisely, a large class of families of g logarithms is closed, and the usual logarithm is self-dual under this map [self-duality here means $\log^*(x) = \log(x)$]. (Without loss of generality, we can set $c = 1$ because c trivially rescales the escort parameter ν.) By requiring that both conditions, Eqs. **31** and **30**, hold—i.e., by inserting Eq. **31** into Eq. **30**—it is possible to derive explicitly a generic form of g logarithms

$$\Lambda(x) = \left[\frac{1}{\frac{2\lambda}{\nu} h[\frac{\nu}{2\lambda}\log(x)]} - \frac{\nu}{2}\right]^{-1}, \qquad [32]$$

with $\lambda \in (0,1]$ and $h(x)$ a monotonically increasing function h: $[-\infty,\infty] \to [-1,1]$, with $h(0) = 0$, $h'(0) = 1$, $h(x) = -h(-x)$, and $\lim_{x\to\infty} h(x) = 1$. Each pair h and λ defines a family of g logarithms parametrized by ν; each such family Λ_ν has the properties (*i*) $\Lambda^*_\nu = \Lambda_{-\nu}$ and (*ii*) $\lim_{\nu\to 0} \Lambda_\nu = \log$. The first property states that the g logarithms associated with the TS and HT entropies are dual to each other. The second property states that in the $\nu \to 0$ limit all these families reproduce the BG case.[§] This result is quite general: If s_{HT} is associated with a g logarithm $\Lambda_{\nu'}$ of the form Eq. **32** for some value ν', then s_{TS} is associated with $\Lambda_{-\nu'}$, with ν' fixing the value of ν in the escort probability.

The specific choice of $h(x) = \tanh(x)$ and $\lambda = 1$ yields the family of q logarithms

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[‡] The duality * acts on pairs (Λ,ν) in such a way that $(\Lambda,\nu)^* = (\Lambda^*,\nu^*)$ with Λ^* as in Eq. **30** and $\nu^* = -\nu$. All elements $(\Lambda,0)$ are self-dual, i.e., $(\Lambda,0)^* = (\Lambda,0)$, allowing us to explore the HT-TS duality in many different directions that go beyond the scope of this paper.

[§] Eq. **31** as the starting point for producing *-closed families of g logarithms with properties (*i*) and (*ii*) is not the most general Ansatz. Other possibilities will be discussed elsewhere.

$$\Lambda(x) = \left[\frac{1}{\frac{2}{\nu}\tanh[\frac{\nu}{2}\log(x)]} - \frac{\nu}{2}\right]^{-1} = \frac{x^\nu - 1}{\nu} = \log_q(x), \qquad [33]$$

where $\log_q(x) = (x^{1-q} - 1)/(1-q)$ and $\nu = 1-q$. We see in a concrete example how ν is uniquely determined by the parametrization (here q) of the family of given distribution functions. It is remarkable how requiring both Eqs. **30** and **31** automatically reproduces the condition necessary to recover Tsallis entropy (19) within the TS approach.[¶] Moreover, the duality $\Lambda_\nu^* = \Lambda_{-\nu}$ written in terms of q logarithms is just the well-known duality $\log_q^* = \log_{2-q}$. The associated HT and TS entropies are

$$\begin{aligned} S_{\rm TS}[p] &= -\sum_i \int_0^{p_i} dx \log_{2-q}\left(\frac{x}{x_0}\right) \\ &= \frac{1 - \sum_i p_i^q}{q-1}, \\ S_{\rm HT}[p] &= -\sum_i \int_0^{p_i} dx \log_q\left(\frac{x}{x_0}\right) \\ &= -\frac{1}{q(2-q)}\sum_i p_i \log_q(p_i) - \frac{1-q}{q(2-q)}, \end{aligned} \qquad [34]$$

with $x_0 = q^{1/(1-q)}$ defined as above by $s'_{\rm TS}(x)|_{x=x_0} = 0$. Clearly $S_{\rm TS}$ gives the Tsallis entropy (20), whereas $S_{\rm HT}$ corresponds to the entropy for power laws discussed in refs. 13 and 14.

State Changes—the Superstatistical Transformation Group

As we have seen before, Eq. **23** determines how the superstatistical Boltzmann factor B and the kernel f transform under state changes $\sigma \to \sigma'$. These transformations are

$$Z^{(\sigma')} = \frac{1}{z}Z^{(\sigma)} \quad \text{and} \quad B^{(\sigma')}(\epsilon) = \frac{1}{z}B^{(\sigma)}(a + b\epsilon), \qquad [35]$$

where z is a ratio of two normalization constants for σ and σ', respectively. For the transformation law of the superstatistical distribution function, we find

$$\begin{aligned} \frac{1}{z}B^{(\sigma)}(a+b\epsilon) &= \frac{1}{z}\int_0^\infty d\beta f^{(\sigma)}(\beta) e^{-\beta(a+b\epsilon)} \\ &= \int_0^\infty d\beta' \left[\frac{1}{z}\frac{1}{b} f^{(\sigma)}\left(\frac{\beta'}{b}\right)e^{-\beta'\frac{a}{b}}\right] e^{-\beta'\epsilon} \\ &= \int_0^\infty d\beta' f^{(\sigma')}(\beta') e^{-\beta'\epsilon} \\ &= B^{(\sigma')}(\epsilon), \end{aligned} \qquad [36]$$

where we substituted $\beta' = b\beta$ in the second line and used Eq. **35** in the third line, together with Eq. **1**. By comparison, f transforms according to

$$f^{(\sigma')}(\beta') = \frac{1}{z}\frac{1}{b} f^{(\sigma)}\left(\frac{\beta'}{b}\right)e^{-\beta'\frac{a}{b}}. \qquad [37]$$

The value of z is fixed by the normalization condition for the kernel, $\int_0^\infty d\beta f^{(\sigma')}(\beta) = 1$.

[¶]If a q exponential is observed in the superstatistical system for some particular value of q, then the entropy reconstruction in the TS approach produces different entropies $s_{\rm TS}$ depending on the choice of ν in the TS-MEP. Each of these entropies can be used to recover the power law (q-exponential distribution) one has used to construct the entropy. Yet only for the particular choice $\nu = 1-q$ does one get $\sum_i s_{\rm TS}(p_i)$ as the Tsallis entropy $S_q[\mathbf{p}]$ for the particular case (10).

For describing the transformation group, we introduce the following notation: State σ is characterized by $a = 0$ and $b = 1$. For state σ', a and b take other values. We now indicate the parameters (a,b) in the kernel through

$$f^{(\sigma)} \equiv f^{(0,1)}, \qquad f^{(\sigma')} \equiv f^{(a,b)}. \qquad [38]$$

Eq. **37** can now be written as

$$f^{(a,b)}(\beta') = \frac{1}{z}\frac{1}{b} f^{(0,1)}\left(\frac{\beta'}{b}\right)e^{-\beta'\frac{a}{b}}. \qquad [39]$$

From Eq. **36**, we see that, under the state change $\sigma \to \sigma'$, the energy ϵ undergoes the affine transformation $\epsilon \to a + b\epsilon$. We define three operators Φ_1, Φ_2, and Π representing translation of ϵ, dilatation of ϵ, and normalization of f, respectively,

$$\begin{aligned} &\Phi_1(a)f(\beta) = e^{-\beta a}f(\beta), \qquad \Phi_2(b)f(\beta) = \frac{1}{b}f\left(\frac{\beta}{b}\right), \\ &\Pi f(\beta) = \frac{f(\beta)}{\int_0^\infty d\beta' f(\beta')}. \end{aligned} \qquad [40]$$

Φ_1 and Φ_2 form groups

$$\Phi_1(a)\Phi_1(a') = \Phi_1(a+a') \qquad \Phi_2(b)\Phi_2(b') = \Phi_2(bb'), \qquad [41]$$

with the identity elements $\Phi_1(0)$ and $\Phi_2(1)$, and the inverse elements

$$\Phi_1^{-1}(a) = \Phi_1(-a), \qquad \Phi_2^{-1}(b) = \Phi_2\left(\frac{1}{b}\right). \qquad [42]$$

$\Phi_1(a)$ and $\Phi_2(b)$ do not commute:

$$\Phi_2(b)\Phi_1(a) = \Phi_1\left(\frac{a}{b}\right)\Phi_2(b), \qquad [43]$$

whereas the projection operator Π commutes with Φ_2. We finally define the group

$$G(a,b) = \Pi\Phi_2(b)\Phi_1(a), \qquad [44]$$

with the following group-composition rule and inverse element:

$$G(a,b)G(a',b') = G(a' + ab', bb'), \qquad G^{-1}(a,b) = G\left(-\frac{a}{b}, \frac{1}{b}\right). \qquad [45]$$

Eq. **39** can now be written as

$$f^{(a,b)}(\beta) = G(a,b)f^{(0,1)}(\beta), \qquad [46]$$

or, more generally,

$$f^{(a',b')}(\beta) = G\left(\frac{a'-a}{b}, \frac{b'}{b}\right)f^{(a,b)}(\beta). \qquad [47]$$

The generators of Φ_1 and Φ_2 are given by $g_1 = -\beta$ and $g_2 = -(1 + \beta\frac{d}{d\beta})$. We have a representation of the Euclidean group in one dimension.

These transformations apply to infinitely many different fs. For example, if $f^{(0,1)}$ leads to a q-exponential distribution with a specific value of q, under these transformations, all possible $f^{(a,b)}$ will give q exponentials with the same value of q. The functional form of distribution functions is preserved under

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the transformation group. More generally, for every type of superstatistical distribution function, we can identify a representative $f^{(0,1)}$ and call it the "progenitor" function, from which a sequence $\Xi(f)$ of $f^{(a,b)}$ can be generated by applying the transformation group. For every type of function f, there exists a unique sequence to which it belongs. These sequences define equivalence classes for the superstatistical kernels. Suppose experiments on a superstatistical system produce a set of functions $\Omega = \{f_{\sigma_i}\}_{i=1}^{N}$ for N different states σ_i, arbitrarily chosen. If $\Omega \subset \Xi(f)$ for some fixed sequence $\Xi(f)$, then state changes of the system transform consistently under the transformation group Eq. **44** and an entropy $S = \sum_i s(p_i)$ exists such that s is invariant under state changes of the superstatistical system.

Discussion

By studying the possibilities of reconstructing entropies from superstatistical distribution functions, we found that there exist only two distinct frameworks, the HT approach (using only normal constraints) and the TS approach (employing escort constraints). A duality between the two approaches is shown to exist, indicating that superstatistical systems can be explored in two ways that are *dual* to each other. Under certain circumstances, the duality allows us to determine the possibilities of functional forms of generalized logarithms and thus of entropic forms, as explicitly shown for systems with characteristic power laws. For systems governed by a particular distribution function, it has to be decided empirically which of the two frameworks applies; these systems may differ considerably in terms of their thermodynamic properties. We studied how the superstatistical kernel transforms under macroscopic state changes within the two approaches. We presented the respective transformation laws and identified the corresponding transformation group to be the Euclidean group in one dimension, which indicates the existence of a remarkably simple mathematical structure behind thermodynamical state changes of superstatistical systems governed by an invariant entropic form.

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